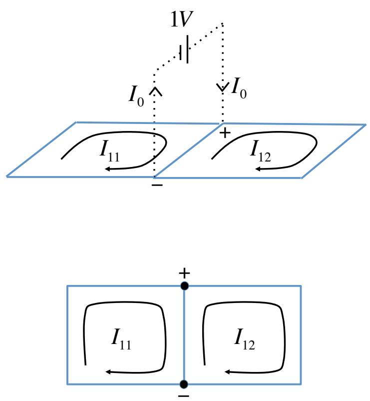
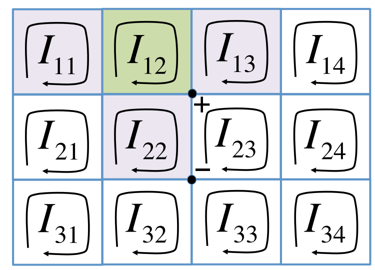
1 General case: 2D grid

We now define the sequence of finite grids we will solve and how to label the variables to simplify coding. Let $k = 1, 2, \ldots,$ denote the sequence number of each grid.

Case $k = 1$ consists of 2 cells, as shown in Figure 1(a), for which the equivalent resistance has been explicitly solved to be 0.6 ohm. To grow the size of the grid symmetrically around nodes $+$ and $-$, Case



(a) Case 1: 2 cells



(b) Case 2: 12 cells

Figure 1: (a) Case $k = 1$: The definition of loop currents is in the upper panel. It is represented in the lower panel with the understanding that loop currents are always in the clockwise direction and that the loop involving the voltage source is not shown. (b) Case $k = 2$: 3×4 loops. KVL around (green) cell $(1, 2)$ involves variables in adjacent (grey) cells: $4I_{12} - I_{11} - I_{13} - I_{22} = 0$.

$k = 2$ has 2 more rows and 2 more columns for a total of 12 cells; see Figure 1(b) and its caption. The variable is the column vector $I := (I_{11}, \ldots, I_{21}, \ldots, I_{34})$ obtained by stacking all the 12 loop currents I_{rc}.

For the general Case k, there are a total of $K := 2k(2k-1)$ current variables $I := (I_{rc},\ r = 1, \ldots, 2k-1,\ c = 1, \ldots, 2k)$. Solving this case numerically boils down to constructing algorithmically the matrix A and the vector b that form the linear equation in the loop current I:

$$A I \quad = \quad b$$

Here, b is a column vector of size K, whose entries are all 0, except $b_{K/2} = -I_0$ and $b_{K/2+1} = I_0$.

The matrix A is $K \times K$. Its diagonal entries are $A_{ii} = 4$, $i = 1, \ldots, K$. For each row, between 2 and 4 off-diagonal entries are -1 and the rest are 0. To determine which off-diagonal entries are -1, the key is to map each cell (r, c) in Figure 1(b) to its position in vector I that has $K = 2k(2k-1)$ components. This mapping is:

$$(r, c) \quad \to \quad (r-1)(2k) \; + \; c$$

For example, the current I_{11} is the first component of the vector I, I_{12} is the second component of I, and so on. Hence A's nonzero entries are given by: for each cell (r, c), $r = 1, \ldots, 2k-1$, $c = 1, \ldots, 2k$,

1. $pos \; := \; 2k(r-1) + c.$
2. $A_{pos,pos} \; := \; 4.$

3. $A_{pos,pos'} := -1$ where pos' is given by $pos' := 2k(i-1)+j$ for

$$(i,j) \quad := \quad \begin{cases} (r,c-1) & \text{if } c>1 \\ (r-1,c) & \text{if } r>1 \\ (r,c+1) & \text{if } c<2k \\ (r+1,c) & \text{if } r<2k-1 \end{cases}$$

The matrix A is nonsingular. Inverting A yields $I_{K/2}$ and $I_{K/2+1}$ in terms of I_0. Applying KVL to the loop involving the voltage source, we have

$$1 \quad = \quad I_0 + I_{K/2} - I_{K/2+1}$$

Combining these yields the equivalent resistance $R_k := 1/I_0$.

Example: Case $k=2$*.* Applying KVL to each of the 12 cells in the grid yields the linear equation

$$\begin{bmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} I_{11} \\ I_{12} \\ I_{13} \\ I_{14} \\ I_{21} \\ \vdots \\ I_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot I_0$$

Inverting the matrix yields $I_{22} = -0.2366\, I_0$ and $I_{23} = 0.2366\, I_0$. Applying KVL around the loop involving the voltage source gives

$$1 \quad = \quad I_0 \;+\; I_{22} \;-\; I_{23}$$

Substituting I_{22} and I_{23}, we have the equivalent resistance R_k for $k=2$:

$$R_k \quad := \quad \frac{1}{I_0} \;=\; 1 - 0.2366 - 0.2366 \;=\; 0.5269 \;\text{ ohm}$$